Brojni nizovi

Brojni niz je realna f-ja definisana nad skupom prirodnih brojeva.

Npr.

$1, 2, 3, \dots, n, n+1, \dots$ je niz prirodnih brojeva. Opšti član ovog niza je $a_n = n$, $n \in \mathbb{N}$. Niz možemo pisati i u obliku $\{n\}_{n\in\mathbb{N}}$.

$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots$ je niz sa opštim članom $b_n = \frac{1}{n}$, $n \in \mathbb{N}$. Ovaj niz možemo pisati i u obliku $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$

$-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \dots$ je niz čiji je opšti član $s_n = \frac{(-1)^n}{n^2}$, $n \in \mathbb{N}$

Skraćeno niz možemo pisati kao $\left\{\frac{(-1)^n}{n^2}\right\}_{n\in\mathbb{N}}$

$\frac{1}{2}, -1, \frac{3}{2}, -2, \frac{5}{2}, -3, \dots$ je niz čiji je opšti član $t_n = \frac{(-1)^{n-1}\cdot n}{2}$

Niz možemo pisati u obliku $\left\{\frac{(-1)^{n-1}}{2}\cdot n\right\}_{n\in\mathbb{N}}$

Aritmetički niz

Aritmetički niz je niz brojeva kod kojih je razlika između dva susjedna člana stalan broj.

$$a_1, a_2, a_3, a_4, \dots, a_n, a_{n+1}, \dots$$

$$\begin{aligned}
&a_2 - a_1 = d \\
&a_3 - a_2 = d \\
&a_4 - a_3 = d \\
&\vdots \\
&a_n - a_{n-1} = d \\
&\vdots
\end{aligned}
\qquad
\begin{aligned}
&a_1 \\
&a_2 = a_1 + d \\
&a_3 = a_2 + d = a_1 + 2d \\
&a_4 = a_3 + d = a_1 + 3d \\
&\vdots \\
&a_n = a_{n-1} + d = a_1 + (n-1)d \\
&\vdots
\end{aligned}$$

$s + t = n + 1$

$$\begin{aligned}
a_s + a_t &= a_1 + (s-1)d + a_1 + (t-1)d = \\
&= 2a_1 + (s+t-2)d = 2a_1 + (n-1)d = \\
&= a_1 + a_n
\end{aligned}$$

$$\begin{aligned}
S_n &= a_1 + a_2 + \dots + a_n \\
+\quad S_n &= a_n + a_{n-1} + \dots + a_1 \\
\hline
2S_n &= (a_1 + a_n) + (a_2 + a_{n-1}) + \dots + (a_n + a_1)
\end{aligned}$$

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d)$$

suma prvih n članova

1. Izračunati sumu prvih 20 članova niza $2, 5, 8, 11, 14, \dots$

Rj. Ovo je aritmetički niz, $d = 3$

$$a_{20} = a_{19} + 3 = a_1 + 19\cdot 3 = 2 + 57 = 59$$

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{20}{2}(2 + 59) = 10\cdot 61 = 610$$ suma prvih dvadeset članova

Geometrijski niz

Geometrijski niz je niz brojeva kod kojeg je količnik dva susjedna člana stalan broj.

$$b_1,\ b_2,\ b_3,\ b_4,\ \dots\ b_{n-1},\ b_n,\ \dots$$

$b_2 : b_1 = q$

$b_3 : b_2 = q$

$b_4 : b_3 = q$

$\vdots$

$b_n : b_{n-1} = q$

$\vdots$

b_1

$b_2 = b_1 q$

$b_3 = b_2 q = b_1 q^2$

$b_4 = b_3 q = b_1 q^3$

$\vdots$

$b_n = b_{n-1} q = b_1 q^{n-1}$

$\vdots$

$S_n = b_1 + b_2 + b_3 + \dots + b_n$

$S_n = b_1 + b_1 q + b_1 q^2 + \dots + b_1 q^{n-1}$

$S_n = b_1(1 + q + q^2 + \dots + q^{n-1}) \;/ (1-q)$

$(1-q)S_n = b_1(1-q)(1+q+q^2+\dots+q^{n-1})$

$(1-q)S_n = b_1(1-q^n) \quad /:(1-q)$

$S_n = b_1 \dfrac{1-q^n}{1-q}$ suma prvih n članova

(2.) Izračunati sumu prvih 50 članova niza $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$

Rj. Ovo je geometrijski niz. $b_1 = \frac{1}{3}$, $q = \frac{1}{3}$, $S_n = b_1 \dfrac{1-q^n}{1-q}$.

$$S_{50} = \frac{1}{3} \cdot \frac{1-(\frac{1}{3})^{50}}{1-\frac{1}{3}} = \frac{1}{3}\cdot\frac{3}{2}\cdot\left(1-\frac{1}{3^{50}}\right) = \frac{1}{2}\left(1-\frac{1}{3^{50}}\right) = \frac{1}{2} - \frac{1}{2\cdot 3^{50}} \approx \frac{1}{2}$$

Monotoni nizovi

Ako je $x_n < x_{n+1}$ tada niz $\{x_n\}_{n\in\mathbb{N}}$ raste

$x_n \le x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ ne opada

$x_n > x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ opada

$x_n \ge x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ ne raste

ove nizove jednim imenom zovemo monotoni nizovi

$$a_{n+1} - a_n = \dots \begin{cases} <0, & \text{niz opada} \\ >0, & \text{niz raste} \end{cases} \qquad \frac{a_{n+1}}{a_n} = \dots \begin{cases} >1, & \text{rastući niz} \\ <1, & \text{opadajući niz} \end{cases}$$

(3.) Ispitati monotonost niza $\{a_n\}_{n\in\mathbb{N}}$ gdje je $a_n = \dfrac{n-1}{2n+1}$.

Rj. $$a_{n+1} - a_n = \frac{n+1-1}{2(n+1)+1} - \frac{n-1}{2n+1} = \frac{n}{2n+3} - \frac{n-1}{2n+1} = \frac{2n^2+n-(2n^2-2n+3n-3)}{(2n+3)(2n+1)}$$

$$= \frac{3}{(2n+3)(2n+1)} > 0,\ \forall n \Rightarrow \{a_n\} \text{ je rastući niz}$$

Granična vrijednost niza

Broj A nazivamo granična vrijednost niza ili limesom niza realnih brojeva $x_1, x_2, \dots, x_n, \dots$, što simbolički pišemo

$$\lim_{n\to\infty} x_n = A$$

ako za svaki $\varepsilon > 0$ postoji broj N (koji zavisi od ε) tako da

$$|x_n - A| < \varepsilon \quad \text{za svaki } n > N.$$

1. Dat je niz $1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots$ Izračunati za koju vrijednost n će biti zadovoljena nejednakost $\frac{1}{n^2} < \varepsilon$ ako je $\varepsilon = 0{,}001$.

Rj. $\frac{1}{n^2} < 0{,}001$

$\frac{1}{n^2} < 10^{-3} \quad / \cdot n^2$

$10^{-3} n^2 > 1 \quad / \cdot 10^3$

$n^2 > 10^3$

$n > 10\sqrt{10} \approx 31{,}62$

Za sve $n > 31$ će biti zadovoljena nejednakost $\frac{1}{n^2} < \varepsilon$.

2. Pokazati da je $\lim_{n\to\infty} \frac{2n+1}{n+1} = 2$.

Rj. Iz definicije $\forall \varepsilon > 0 \ \exists N$ (koji zavisi od ε) tako da

$$\left|\frac{2n+1}{n+1} - 2\right| < \varepsilon \quad \text{za svaki } n > N.$$

$$\left|\frac{2n+1}{n+1} - 2\right| = \left|\frac{2n+1-2n-2}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \varepsilon$$

$(n+1)\varepsilon > 1 \quad / : \varepsilon \ (\varepsilon > 0$

$n+1 > \frac{1}{\varepsilon}$

$n > \frac{1}{\varepsilon} - 1$

Prema tome za svaki pozitivan broj ε postoji takav broj N $\left(N = \frac{1}{\varepsilon} - 1\right)$ takav da za $n > N$ vrijedi $\left|\frac{2n+1}{n+1} - 2\right| < \varepsilon$.

Prema tome $\lim_{n\to\infty} \frac{2n+1}{n+1} = 2$.

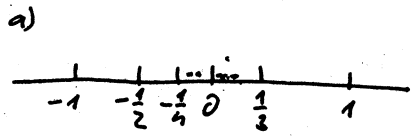
3. Odredite limese nizova

a) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, \frac{(-1)^{n-1}}{n}, \dots$;

b) $\frac{2}{1}, \frac{4}{3}, \frac{6}{5}, \dots, \frac{2n}{2n-1}, \dots$;

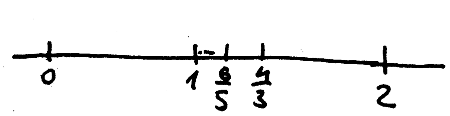
c) $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$;

Rj. a)



0

b)



1

c) $\sqrt{2} \approx 1{,}41$

$\sqrt{2\sqrt{2}} = \sqrt[4]{8} \approx 1{,}68$

$\sqrt{2\sqrt{2\sqrt{2}}} = \sqrt{2\sqrt[4]{8}} = \sqrt[8]{2^7} \approx 1{,}83$

$\sqrt{2}, \sqrt[4]{2^3}, \sqrt[8]{2^7}, \ldots, \sqrt[2^n]{2^{2^n-1}}, \qquad \lim_{n\to\infty} 2^{\frac{2^n-1}{2^n}} = 1$

Operacije sa limesima

a) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

b) $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

d) $\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{\lim_{n\to\infty} a_n}$

e) $\lim_{n\to\infty} b^{a_n} = b^{\lim_{n\to\infty} a_n}, \quad b>0$

f) $\lim_{n\to\infty} \log_b a_n = \log_b \lim_{n\to\infty} a_n, \quad b>1$

1. Izračunajte limese

a) $\lim_{n\to\infty} \frac{1}{n}$ Rj. $\lim_{n\to\infty} \frac{1}{n} = 0$

b) $\lim_{n\to\infty} 7$ Rj. $\lim_{n\to\infty} 7 = 7$

c) $\lim_{n\to\infty} n^2$ Rj. $\lim_{n\to\infty} n^2 = \infty$

d) $\lim_{n\to\infty} \frac{n}{n+1}$

Rj. $\lim_{n\to\infty} \frac{n}{n+1} \left(=\frac{\infty}{\infty}\right) \begin{matrix} /:n \\ /:n \end{matrix} = \lim_{n\to\infty} \frac{1}{1+\underbrace{\frac{1}{n}}_{\to 0}} = 1$

e) $\lim_{n\to\infty} \frac{n^2+n-3}{n^3+n^2+1} \begin{matrix} /:n^3 \\ /:n^3 \end{matrix}$ Rj. 0

Neodređeni izrazi su $\frac{0}{0}, \infty-\infty, 0\cdot\infty, \frac{\infty}{\infty}, \frac{\infty}{0}$

Određeni izrazi su $\infty\cdot\infty = \infty, \infty+\infty = \infty, \frac{0}{\infty} = 0$

2. Izračunati limese:

a) $\lim_{n\to\infty} \frac{n^3+3n+9}{2n^2+3n-1}$ Rj. $\lim_{n\to\infty} \frac{n^3+3n+9}{2n^2+3n-1} \begin{matrix} /:n^3 \\ /:n^3 \end{matrix} = \lim_{n\to\infty} \frac{1+\overbrace{\frac{3}{n^2}}^{\to 0}+\overbrace{\frac{9}{n^3}}^{\to 0}}{\underbrace{\frac{2}{n}}_{\to 0}+\underbrace{\frac{3}{n^2}}_{\to 0}-\underbrace{\frac{1}{n^3}}_{\to 0}} = \frac{1}{0} = \infty$

b) $\lim_{n\to\infty} \frac{n^2+2n+3}{2n^2+n-4}$ Rj. $\lim_{n\to\infty} \frac{n^2+2n+3}{2n^2+n-4} \begin{matrix} /:n^2 \\ /:n^2 \end{matrix} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{3}{n^2}}{2+\frac{1}{n}-\frac{4}{n^2}} = \frac{1}{2}$

c) $\lim_{n\to\infty} \frac{3n^3+n-1}{2n^4+1}$ Rj. $\lim_{n\to\infty} \frac{3n^3+n-1}{2n^4+1} \begin{matrix} /:n^4 \\ /:n^4 \end{matrix} = \lim_{n\to\infty} \frac{\frac{3}{n}+\frac{1}{n^3}-\frac{1}{n^4}}{2+\frac{1}{n^4}} = \frac{0}{2} = 0$

d) $\lim_{n\to\infty} \frac{(n+1)(n+2)(n+3)}{n^3}$ Rj. $\lim_{n\to\infty} \frac{(n+1)(n+2)(n+3)}{n^3} \begin{matrix} /:n^3 \\ /:n^3 \end{matrix} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{2}{n})(1+\frac{3}{n})}{1} = \frac{1}{1} = 1$

e) $\lim_{n\to\infty} \frac{n+(-1)^n}{3n-(-1)^n}$ Rj. $\lim_{n\to\infty} \frac{n+(-1)^n}{3n-(-1)^n} \begin{matrix} /:n \\ /:n \end{matrix} = \lim_{n\to\infty} \frac{1+\overbrace{\frac{(-1)^n}{n}}^{\to 0}}{3-\underbrace{\left(\frac{(-1)^n}{n}\right)}_{\to 0}} = \frac{1}{3}$

3. Izračunati limese:

a) $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}\right)$

b) $\lim\limits_{n\to\infty}\left(\frac{1+3+5+\ldots+(2n-1)}{n+1}-\frac{2n+1}{2}\right)$

c) $\lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}\right)$

d) $\lim\limits_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\ldots+\frac{(-1)^{n-1}}{3^{n-1}}\right)$

Rj. a) rj. $\frac{1}{2}$ c) Rj. 1

b) $\lim\limits_{n\to\infty}\left(\frac{1+3+5+\ldots+(2n-1)}{n+1}-\frac{2n+1}{2}\right)=\lim\limits_{n\to\infty}\left(\frac{\frac{n}{2}(1+2n-1)}{n+1}-\frac{2n+1}{2}\right)=\lim\limits_{n\to\infty}\left(\frac{2n^2}{2n+2}-\frac{2n+1}{2}\right)$

$=\lim\limits_{n\to\infty}\frac{2n^2-\overbrace{(2n+1)(n+1)}^{2n^2+3n+1}}{2(n+1)}=\lim\limits_{n\to\infty}\frac{\cancel{2n^2}-\cancel{2n^2}-3n-1}{2n+2}\begin{matrix}/:n\\ /:n\end{matrix}=\lim\limits_{n\to\infty}\frac{-3-\frac{1}{n}}{2+\frac{2}{n}}=-\frac{3}{2}$

d) imamo niz $1,-\frac{1}{3},\frac{1}{9},-\frac{1}{27},\ldots$ količnik dva susjedna člana je $-\frac{1}{3}$

imamo geometrijski niz, $|q|<1$, $S_n=a_1\frac{1-q^n}{1-q}$

$\lim\limits_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{9}-\ldots+\frac{(-1)^{n-1}}{3^{n-1}}\right)=\lim\limits_{n\to\infty}\left(1\cdot\frac{1-\left(-\frac{1}{3}\right)^n\to 0}{1-\left(-\frac{1}{3}\right)}\right)=\frac{1}{1+\frac{1}{3}}=\frac{1}{\frac{4}{3}}=\frac{3}{4}$

4. Izračunati limese:

a) $\lim\limits_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{2^n+3^n}$ b) $\lim\limits_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$ c) $\lim\limits_{n\to\infty}\frac{n\sin n!}{n^2+1}$

d) $\lim\limits_{x\to\infty}\frac{(2x-3)(3x+5)(4x-6)}{3x^3+x-1}$ e) $\lim\limits_{x\to\infty}\frac{1000x}{x^2-1}$ f) $\lim\limits_{x\to\infty}\frac{2x^2-x^3-4}{\sqrt{x^4+1}}$

g) $\lim\limits_{x\to\infty}\frac{2x+3}{x+\sqrt[3]{x}}$ h) $\lim\limits_{x\to\infty}\frac{x^2}{10+x\sqrt{x}}$ i) $\lim\limits_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$

Rj. a) $\lim\limits_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\begin{matrix}/:3^n\\ /:3^n\end{matrix}=\lim\limits_{n\to\infty}\frac{\frac{2^{n+1}}{3^n}+3}{\left(\frac{2}{3}\right)^n+1}=3$

b) $\lim\limits_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\lim\limits_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\infty}=0$

i) $\lim\limits_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}=\lim\limits_{x\to+\infty}\left(\frac{x}{x+\sqrt{x+\sqrt{x}}}\right)^{\frac{1}{2}}=\lim\limits_{x\to+\infty}\left(\frac{1}{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}\right)^{\frac{1}{2}}=1$

c) rj. 0 d) rj. 8 e) rj. 0 f) rj. 2 g) rj. 2 h) ∞

Granična vrijednost f-je

Kažemo da f-ja $f(x) \to A$ kada $x \to p$ (A i p su brojevi) ili da je $\lim\limits_{x\to a} f(x) = A$ ako za svaki $\varepsilon > 0$ postoji takav $\delta > 0$ (δ zavisi od ε) da je $|f(x) - A| < \varepsilon$ za $0 < |x-p| < \delta$.

1. Izračunati limese:

a) $$\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2} \left(=\frac{0}{0}\right) = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+2}{x-1} = 4$$

b) $$\lim_{x\to -1} \frac{x^3+1}{x^2+1} = \frac{0}{2} = 0$$

c) $$\lim_{x\to 5} \frac{x^2-5x+10}{x^2-25} = \frac{25-25+10}{0} = \infty$$

d) $$\lim_{x\to -1} \frac{x^2-1}{x^2+3x+2} \left(=\frac{0}{0}\right) = \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)(x+2)} = \lim_{x\to -1} \frac{x-1}{x+2} = \frac{-2}{1} = -2$$

e) $$\lim_{x\to 2} \frac{x^2-2x}{x^2-4x+4} \left(=\frac{0}{0}\right) = \lim_{x\to 2} \frac{x(x-2)}{(x-2)^2} = \lim_{x\to 2} \frac{x}{x-2} = \frac{2}{0} = +\infty$$

f)$^{\vee}$ $$\lim_{x\to 1} \frac{x^3-3x+2}{x^4-4x+3}$$ Rj. $\frac{1}{2}$

g) $$\lim_{x\to a} \frac{x^2-(a+1)x+a}{x^3-a^3} \left(=\frac{0}{0}\right) = \lim_{x\to a} \frac{(x-a)(x-1)}{(x-a)(x^2+ax+a^2)} = \frac{a-1}{a^2+a^2+a^2} = \frac{a-1}{3a^2}$$

$$\begin{array}{l}
(x^2 \overbrace{-(a+1)x}^{-ax-x} + a) : (x-a) = x-1 \\
\underline{-\ x^2 - ax} \\
\qquad -x+a \\
\qquad \underline{-x+a} \\
\qquad /\!/ \quad /\!/
\end{array}$$

$$\begin{array}{c}
1 \\
1\ 1 \\
1\ 2\ 1 \\
1\ 3\ 3\ 1
\end{array}$$

h) $$\lim_{h\to 0} \frac{(x+h)^3-x^3}{h} \left(=\frac{0}{0}\right) = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2$$

i)$^{\vee}$ $$\lim_{x\to 1}\left(\frac{1}{1-x} - \frac{3}{1-x^3}\right)$$ Rj. -1.

2. Izračunati limese

a) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}\left(=\dfrac{0}{0}\right)=\left|\begin{matrix}\text{uvodimo smjenu}\\ 1+x=y^6\\ x\to 0\Rightarrow y\to 1\end{matrix}\right|=\lim\limits_{y\to 1}\dfrac{y^3-1}{y^2-1}=\lim\limits_{y\to 1}\dfrac{(y-1)(y^2+y+1)}{(y-1)(y+1)}=\dfrac{3}{2}$

b) $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}\left(=\dfrac{0}{0}\right)=\left|\begin{matrix}x=t^2\\ x\to 1\Rightarrow t\to 1\end{matrix}\right|=\lim\limits_{t\to 1}\dfrac{t-1}{t^2-1}=\lim\limits_{t\to 1}\dfrac{t-1}{(t-1)(t+1)}=\dfrac{1}{2}$

c) $\lim\limits_{x\to 64}\dfrac{\sqrt{x}-8}{\sqrt[3]{x}-4}$ Rj. 3

d) $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}\left(=\dfrac{0}{0}\right)=\left|\begin{matrix}x=t^{12}\\ x\to 1\Rightarrow t\to 1\end{matrix}\right|=\lim\limits_{t\to 1}\dfrac{\overbrace{t^4-1}^{(t^2-1)(t^2+1)}}{t^3-1}=\lim\limits_{t\to 1}\dfrac{\cancel{(t-1)}(t+1)(t^2+1)}{\cancel{(t-1)}(t^2+t+1)}=\dfrac{4}{3}$

e) $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$ Rj. $\dfrac{1}{9}$

3. Izračunati limese

a) $\lim\limits_{x\to a}\dfrac{\sqrt{x}-\sqrt{a}}{x-a}\left(=\dfrac{0}{0}\right)=\lim\limits_{x\to a}\dfrac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{(x-a)(\sqrt{x}+\sqrt{a})}=\lim\limits_{x\to a}\dfrac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}=\dfrac{1}{2\sqrt{a}}\ (a>0)$

b) $\lim\limits_{x\to 7}\dfrac{2-\sqrt{x-3}}{x^2-49}\left(=\dfrac{0}{0}\right)=\lim\limits_{x\to 7}\dfrac{(2-\sqrt{x-3})(2+\sqrt{x-3})}{\underbrace{(x^2-49)}_{(x-7)(x+7)}(2+\sqrt{x-3})}=\lim\limits_{x\to 7}\dfrac{\cancel{7-x}}{(-1)\cancel{(7-x)}\underbrace{(x+7)}_{14}\underbrace{(2+\sqrt{x-3})}_{4}}=-\dfrac{1}{56}$

c) $\lim\limits_{x\to 8}\dfrac{x-8}{\sqrt[3]{x}-2}$ Rj. 12

d) $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}\left(=\dfrac{0}{0}\right)=\lim\limits_{x\to 1}\dfrac{(\sqrt{x}-1)(\sqrt{x}+1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)(\sqrt{x}+1)}=\lim\limits_{x\to 1}\dfrac{\cancel{(x-1)}(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{\cancel{(x-1)}(\sqrt{x}+1)}=\dfrac{3}{2}$

e) $\lim\limits_{x\to 4}\dfrac{3-\sqrt{5+}}{1-\sqrt{5-x}}\left(=\dfrac{0}{0}\right)=\lim\limits_{x\to 4}\dfrac{(3-\sqrt{5+x})(3+\sqrt{5+x})(1+\sqrt{5-x})}{(1-\sqrt{5-x})(1+\sqrt{5-x})(3+\sqrt{5+x})}=\lim\limits_{x\to 4}\dfrac{(4-x)(1+\sqrt{5-x})}{\underbrace{(-4+x)}_{(-1)(4-x)}(3+\sqrt{5+x})}=\dfrac{2}{-6}=-\dfrac{1}{3}$

f) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ Rj. 1

g) $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}\left(=\dfrac{0}{0}\right)=\lim\limits_{h\to 0}\dfrac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h\cdot(\sqrt{x+h}+\sqrt{x})}=\lim\limits_{h\to 0}\dfrac{x+h-x}{h\cdot(\sqrt{x+h}+\sqrt{x})}=\dfrac{1}{2\sqrt{x}}$

h) $\lim\limits_{h\to 0}\dfrac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}\ (x\neq 0)$, Rj. $\dfrac{1}{3\sqrt[3]{x^2}}$

i) $\lim\limits_{x\to 3}\dfrac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$ Rj. $-\dfrac{1}{3}$

4. Izračunati limese

a) $\lim\limits_{x\to+\infty}(\sqrt{x+a}-\sqrt{x})\ (=\infty-\infty)=\lim\limits_{x\to+\infty}\dfrac{(\sqrt{x+a}-\sqrt{x})(\sqrt{x+a}+\sqrt{x})}{(\sqrt{x+a}+\sqrt{x})}=\lim\limits_{x\to+\infty}\dfrac{x+a-x}{(\sqrt{x+a}+\sqrt{x})}=\dfrac{a}{+\infty}=0$

b) $\lim\limits_{x\to+\infty}[\sqrt{x(x+a)}-x]\ (=\infty-\infty)=\lim\limits_{x\to+\infty}\dfrac{[\sqrt{x(x+a)}-x][\sqrt{x(x+a)}+x]}{\sqrt{x(x+a)}+x}=\lim\limits_{x\to+\infty}\dfrac{x^2+ax-x^2}{\sqrt{x(x+a)}+x}=$

$=\lim\limits_{x\to+\infty}\dfrac{ax\ \ /:x}{\sqrt{x(x+a)}+x\ \ /:x}=\lim\limits_{x\to+\infty}\dfrac{a}{\sqrt{1\cdot(1+\underbrace{\frac{a}{x}}_{\to 0})}+1}=\dfrac{a}{2}$

c)$^{\vee}$ $\lim\limits_{x\to+\infty}(\sqrt{x^2-5x+6}-x)$ Rj. $-\frac{5}{2}$

d) $\lim\limits_{x\to+\infty}x(\sqrt{x^2+1}-x)\ (=\infty(\infty-\infty))=\lim\limits_{x\to+\infty}\dfrac{x(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)}=\lim\limits_{x\to+\infty}\dfrac{x(x^2+1-x^2)}{(\sqrt{x^2+1}+x)}$

$=\lim\limits_{x\to\infty}\dfrac{x\ \ /:x}{\sqrt{x^2+1}+x\ \ /:x}=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{1+\underbrace{\frac{1}{x^2}}_{\to 0}}+1}=\dfrac{1}{2}$

e)$^{\vee}$ $\lim\limits_{x\to\infty}(x+\sqrt[3]{1-x^3})$ Rj. 0

Navedimo nekoliko važnih graničnih vrijednosti:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)=e^k \qquad \lim_{n\to\infty}\sqrt[n]{n}=1$$

$$\lim_{n\to\infty}\sqrt[n]{a}=1 \qquad \lim_{n\to\infty}\frac{a^n}{n}=\infty \qquad \lim_{n\to\infty}\frac{n^k}{a^n}=0$$

5. Izračunati limese

a) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x}=\lim\limits_{x\to 0}\left(\dfrac{\sin 5x}{5x}\cdot 5\right)=1\cdot 5=5$

b) $\lim\limits_{x\to 2}\dfrac{\sin x}{x}=\dfrac{1}{2}\sin 2$

c) $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=\left|\begin{array}{l}\text{kako je}\\ -1\le\sin\alpha\le 1\\ \text{za }\forall\alpha\end{array}\right|=0$

d)$^{\vee}$ $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$ Rj. 3

e) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\sin 2x}=\lim\limits_{x\to 0}\dfrac{\frac{\sin 5x}{5x}\cdot 5}{\frac{\sin 2x}{2x}\cdot 2}=\dfrac{5}{2}$

e) $\lim\limits_{x\to\pi} \dfrac{\sin mx}{\sin nx} = \left| \begin{matrix} x = \pi + t \\ x \to \pi \Rightarrow t \to 0 \end{matrix} \right| = \lim\limits_{t\to 0} \dfrac{\sin(m\pi + mt)}{\sin(n\pi + nt)} = \lim\limits_{t\to 0} \dfrac{\overbrace{\sin m\pi \cos mt}^{} + \underbrace{\sin mt \cos m\pi}_{}}{\underbrace{\sin n\pi}_{=0} \cos nt + \sin nt \cos n\pi}$

$= \lim\limits_{t\to 0} \dfrac{(-1)^m \sin mt}{(-1)^n \sin nt} = (-1)^{m-n} \lim\limits_{t\to 0} \dfrac{\frac{\sin mt}{mt}\cdot mt}{\frac{\sin nt}{nt}\cdot nt} = (-1)^{m-n}\dfrac{m}{n}$

f) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \dfrac{2\sin^2\frac{x}{2}}{x^2} = \lim\limits_{x\to 0} \dfrac{2(\sin\frac{x}{2})^2}{4\cdot(\frac{x}{2})^2} = \lim\limits_{x\to 0} \dfrac{1}{2}\left(\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \dfrac{1}{2}$

$$\left.\begin{matrix} 1 = \sin^2 x + \cos^2 x & \quad 1 = \sin^2\frac{x}{2} + \cos^2\frac{x}{2} \\ \cos 2x = \cos^2 x - \sin^2 x & \quad \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \end{matrix}\right\} - \Rightarrow 1 - \cos x = 2\sin^2\frac{x}{2}$$

g) $\lim\limits_{x\to 0} \dfrac{\operatorname{tg} x}{x} = \lim\limits_{x\to 0} \dfrac{\frac{\sin x}{\cos x}}{x} = \lim\limits_{x\to 0} \dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x} = 1$

h)$^\vee$ $\lim\limits_{x\to 1} \dfrac{\sin \pi x}{\sin 3\pi x}$ Rj. $\dfrac{1}{3}$ i)$^\vee$ $\lim\limits_{n\to\infty} \left(n \sin\dfrac{\pi}{n}\right)$ Rj. π

j) $\lim\limits_{x\to 0} \dfrac{\sin 5x - \sin 3x}{\sin x} = \lim\limits_{x\to 0} \dfrac{5\cdot\frac{\sin 5x}{5x} - \frac{\sin 3x}{3x}\cdot 3}{\frac{\sin x}{x}} = 5 - 3 = 2$

k) $\lim\limits_{x\to a} \dfrac{\sin x - \sin a}{x-a} = \lim\limits_{x\to a} \dfrac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a} = \lim\limits_{x\to a} \dfrac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}\cdot\cos\dfrac{x+a}{2} = \cos a$

$$\left.\begin{matrix} \sin x = \sin\left(\frac{x-a}{2} + \frac{x+a}{2}\right) = \sin\frac{x-a}{2}\cos\frac{x+a}{2} + \sin\frac{x+a}{2}\cos\frac{x-a}{2} \\ -\sin a = \sin(-a) = \sin\left(\frac{x-a}{2} - \frac{x+a}{2}\right) = \sin\frac{x-a}{2}\cos\frac{x+a}{2} - \sin\frac{x+a}{2}\cos\frac{x-a}{2} \end{matrix}\right\} +$$

$\sin x - \sin a = 2\sin\dfrac{x-a}{2}\cos\dfrac{x+a}{2}$

6) Izračunati limese

a) $\lim\limits_{x\to\infty} \left(\dfrac{x-1}{x+1}\right)^x = \lim\limits_{x\to\infty} \left(\dfrac{\frac{x-1}{x}}{\frac{x+1}{x}}\right)^x = \lim\limits_{x\to\infty} \dfrac{(1-\frac{1}{x})^x}{(1+\frac{1}{x})^x} = \dfrac{\lim\limits_{x\to\infty}(1+\frac{-1}{x})^x}{e} = \dfrac{e^{-1}}{e} = e^{-2}$

b) $\lim\limits_{x\to 0} \left(\dfrac{2+x}{3-x}\right)^x = \left(\dfrac{2}{3}\right)^0 = 1$

c) $\lim\limits_{x\to\infty} \left(\dfrac{x+1}{2x+1}\right)^{x^2} = \lim\limits_{x\to\infty} \left(\dfrac{1+\frac{1}{x}}{2+\frac{1}{x}}\right)^{x^2} = \left(\dfrac{1}{2}\right)^{+\infty} = 0$

d)$^\vee$ $\lim\limits_{x\to 1} \left(\dfrac{x-1}{x^2-1}\right)^{x+1}$ Rj. $\dfrac{1}{4}$ e) $\lim\limits_{x\to\infty} \left(\dfrac{1}{x^2}\right)^{\frac{2x}{x+1}}$ Rj. 0

Izračunati limes $\lim_{n\to\infty}\left(\frac{1+2+3+\ldots+(n-1)}{n+1}-\frac{n}{2}\right)$

Rj.

$$1+2+3+\ldots+(n-1)=\frac{n-1}{2}(1+(n-1)) \leftarrow \text{suma aritmetičkog niza}$$

$$=\frac{n-1}{2}\cdot n=\frac{n(n-1)}{2}$$

$$\lim_{n\to\infty}\left(\frac{1+2+3+\ldots+(n-1)}{n+1}-\frac{n}{2}\right)=\lim_{n\to\infty}\left(\frac{\frac{n(n-1)}{2}}{n+1}-\frac{n}{2}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{n(n-1)}{2(n+1)}-\frac{n}{2}\right)=\lim_{n\to\infty}\frac{n(n-1)-n(n+1)}{2(n+1)}=$$

$$=\lim_{n\to\infty}\frac{n^2-n-n^2-n}{2n+2}=\lim_{n\to\infty}\frac{-2n}{2(n+1)}=\lim_{n\to\infty}\frac{-n\;:n}{n+1\;:n}\left(=\frac{\infty}{\infty}\right)$$

$$=\lim_{n\to\infty}\frac{-1}{1+\underbrace{\frac{1}{n}}_{0}}=-1$$

Izračunati $\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{1-x}$.

Rj. $a^3-b^3=(a-b)(a^2+ab+b^2)$

$$(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)=(\sqrt[3]{x})^3-1^3=x-1$$

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{1-x}\left(\frac{0}{0}\right)=-\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}\cdot\frac{(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}=-\lim_{x\to 1}\frac{\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}$$

$$=-\lim_{x\to 1}\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=\frac{-1}{\sqrt[3]{1}+\sqrt[3]{1}+1}=-\frac{1}{3}$$

Jednostrani limesi

Ako je $x<a$ i $x\to a$, tada po dogovoru pišemo $x\to a-0$; analogno, ako je $x>a$ i $x\to a$, pišemo to ovako $x\to a+0$.
Brojeve $f(a-0)=\lim\limits_{x\to a-0} f(x)$ i $f(a+0)=\lim\limits_{x\to a+0} f(x)$

nazivamo lijevi limes f-je $f(x)$ u tački a i desni limes f-je $f(x)$ u tački a (ako ti brojevi postoje).
Koriste se i sledeće dvije oznake

$$f(a+)=\lim_{x\to a+} f(x) \quad i \quad f(a-)=\lim_{x\to a-} f(x)$$

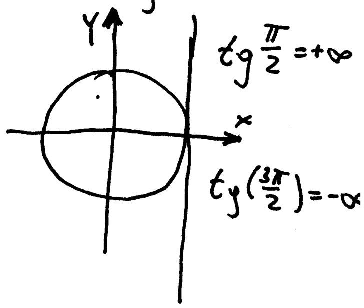
Za postojanje limesa f-je $f(x)$ kada $x\to a$ potrebno je i dovoljno da vrijedi jednakost $f(a-0)=f(a+0)$.

1° Izračunati desni i lijevi limes f-je $f(x)=\operatorname{arc\,tg}\frac{1}{x}$

Rj. $f(+0)=\lim\limits_{x\to +0}\operatorname{arc\,tg}\frac{1}{x}=\frac{\pi}{2}$

$f(-0)=\lim\limits_{x\to -0}\operatorname{arc\,tg}\frac{1}{x}=-\frac{\pi}{2}$

limes f-je $f(x)$ kad $x\to 0$ u ovom slučaju ne postoji



2° Izračunati jednostrane limese

a) $\lim\limits_{x\to -0}\dfrac{1}{1+e^{\frac{1}{x}}}=\dfrac{1}{1+e^{-\infty}}=\dfrac{1}{1+\frac{1}{e^{\infty}}}=1$ b)$^{\vee}$ $\lim\dfrac{1}{1+e^{\frac{1}{x}}}$ Rj. 0

c) $\lim\limits_{x\to 2+0}\dfrac{x}{x-2}=\dfrac{2+0}{2+0-2}=\dfrac{2+0}{+0}=+\infty$ d)$^{\vee}$ $\lim\limits_{x\to 2-0}\dfrac{x}{x-2}$ Rj. $-\infty$

e) $\lim\limits_{x\to -0}\dfrac{|\sin x|}{x}=\lim\limits_{x\to -0}\dfrac{-\sin x}{x}=-1$ f)$^{\vee}$ $\lim\limits_{x\to +0}\dfrac{|\sin x|}{x}$ Rj. 1

g) $\lim\limits_{x\to 1-0}\dfrac{x-1}{|x-1|}=\lim\limits_{x\to 1-0}\dfrac{(x-1)}{-(x-1)}=\lim\limits_{x\to 1-0}(-1)=-1$ h)$^{\vee}$ $\lim\limits_{x\to 1+0}\dfrac{x-1}{|x-1|}$ Rj. 1

i) $\lim\limits_{x\to -\infty}\dfrac{x}{\sqrt{x^2+1}}=\lim\limits_{x\to -\infty}\dfrac{x}{\sqrt{x^2}}=\lim\limits_{x\to -\infty}\dfrac{x}{|x|}=\lim\limits_{x\to -\infty}-\dfrac{x}{x}=-1$ j)$^{\vee}$ $\lim\limits_{x\to +\infty}\dfrac{x}{\sqrt{x^2+1}}$ Rj. 1

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